

BPS States, String Duality, and Nodal Curves on K3

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We describe the counting of BPS states of Type II on K3 by relating the supersymmetric cycles of genus g to the number of rational curves with g double points on K3. The generating function for the number of such curves is the left-moving partition function of the bosonic string

1. Introduction

The revolution afoot in string theory bespeaks the promise of understanding non-perturbative strings. Solitonic states of the low-energy physics [1] have been interpreted as simple conformal field theories of open strings [2]. This observation has had dramatic consequences, most strikingly in helping to provide evidence for string duality, wherein the non-perturbative physics of one theory is equated with the perturbative regime of another [3][4]. The hallmark for demonstrating such equalities has been a counting of BPS states, which are the signatures of a supersymmetric theory. D-branes in Type II string theory, equivalent to the so-called supersymmetric cycles, are precisely such states. The minimum energy states in the presence of a D-brane can be attained by solving for the vacua of the effective low-energy quantum field theory, which is a counting of the number of such BPS states.

This problem, for Type II compactification on K3, has been detailed in a recent paper [5], in which the authors equate this problem to a supersymmetric quantum field theory with target space equal to a certain moduli space. (For the problem on T^4 , see also [6][7][8].) The moduli space is that of a supersymmetric cycle, or in mathematical terms “special Lagrangian manifold,” with a line bundle on K3 – which they argue to be cohomologically equivalent to symmetric products of K3 itself. In this paper, we show that this moduli space problem is equivalent to the mathematical question of counting the number of rational curves in K3 with n double points (where pairs of points are mapped to the same point). We count these by considering the hyperplanes in a projective space \mathbf{P}^n which intersect an embedded K3 at a Riemann surface of genus n . Quite strikingly, the string duality equating Type II on K3 to the heterotic string on T^4 provides a very simple explanation of this counting, yielding a generating function for all such numbers for rational curves of arbitrary topological genus – it is none other than the partition function of the bosonic string! The first six coefficients agree with the current results known by mathematicians [9]; the remaining coefficients can be taken as explicit conjectures for further verification. This counting is currently being pursued [10].

In section two we review the relevant aspects of string theory which lead to the conclusion that special Lagrangian submanifolds of a compactifying space correspond to BPS solutions of the effective supergravity theory, or D-branes. We review the work of Ref. [5], which gives the physical counting of states. The heterotic duality gives an equivalent counting. In section three we describe why this counting is equivalent to the mathematical question described above, and report the results of Ref. [9]. In section four we conclude with some conjectures and prospects for further study.

2. String Theory, BPS States, and D-Branes on K3

The low energy description of Type II string theory is a supergravity theory. Recently, Becker et al. have looked at solitonic solutions of this theory which preserve half the supersymmetries [1].

A supersymmetric three-cycle is defined as a three-cycle solution of eleven dimensional supergravity, compactified on a Calbi-Yau, which preserves one-half of the supersymmetries (which are generically totally broken). This was shown to be a good definition for the Type II theory [1], whose strong coupling behavior is believed to be the eleven dimensional supergravity [3]. The condition leads to the definition that $i : C \hookrightarrow X$ is an embedding of a supersymmetric cycle ($\dim_{\mathbf{R}} C = 3$; $\dim_{\mathbf{R}} X = 6$) if

$$i^* \Omega \sim V, \quad i^* \omega = 0,$$

where Ω is the holomorphic three-form, ω the Kähler class, V the volume form induced from the embedding, and the proportionality is described by a multiple which is constant – not a function – on C . (This constant can be set to one at a point in moduli space by a rescaling of Ω .) The second condition is that of a Lagrangian surface, and the two together describe “special Lagrangian” submanifolds [11]. Unfortunately, although they are closely related to the much-studied minimal submanifolds, very little is known about such submanifolds in non-trivial spaces.

In the present case, we study cycles which have two dimensions in a K3, and the rest flat ($\mathbf{R} \times S^1$). We consider the worldvolume of a spatial three-brane, which will have a four-dimensional effective theory of the low-energy physics. The two-manifold in K3 will be a special Lagrangian submanifold, and such were proven [12] to be equivalent to curves which are holomorphic with respect to one of the complex structures on K3 (see also the argument in [5]).*

The thrust of Ref. [5] was to study the effective theory in the presence of a D-brane. As D-branes are BPS states, they represent minimum-energy states in a given topological sector (those which saturate the Bogomol’nyi bound). It has been argued [13][14] that the low energy field theory limit of n coincident D-brane strings was a $U(n)$ gauge theory reduced to the full dimension of the D –brane (including time). This comes about from the n $U(1)$ ’s of the open string spectra plus the now massless $n(n-1)$ strings stretching between

* Recall that the hyperkahler structure gives us an S^2 of complex structures from the three independent, quaternionic J_i : $J = aJ_1 + bJ_2 + cJ_3$ is a complex structure for $a^2 + b^2 + c^2 = 1$.

different D -branes. As discussed above, we enlarge the space by a circle (converting Type IIA to IIB) so that we may think of the three-brane as a two-surface, Σ , crossed with $S^1 \times \mathbf{R}$, a four-dimensional space. In fact, it is a twisted supersymmetric field theory, precisely of the type considered in Ref. [15], so its ground states can be obtained by reduction from four to two dimensions, yielding a two-dimensional supersymmetric sigma model on $S^1 \times \mathbf{R}$. What is the target space of this sigma model?

As shown in [5] and [15], the target space is the space of solutions to the equations which must be solved for the dimensional reduction to make sense – the Hitchin equations:

$$F_{z\bar{z}} = [\Phi_z, \Phi_{\bar{z}}]$$

$$D_z \Phi_{\bar{z}} = 0 = D_{\bar{z}} \Phi_z.$$

Here F is the gauge field strength and Φ is an adjoint-valued one-form – identified with the normal vector in the dimensional reduction of the gauge field, which is possible due to the codimension one embedding in a space with trivial canonical bundle, the K3. For our purposes, we will only consider one D-brane ($n = 1$), so that $F = 0$ and we have the space of flat connections and a choice of harmonic one-form. The one-form describes motions in the normal direction, so is equivalently giving a local parametrization of the space of holomorphic surfaces of genus g . The “compactification” of the space of solutions to the Hitchin equations then yields a moduli space, \mathcal{M}_g^H , describing a choice of holomorphic Riemann surface in K3 and a flat $U(1)$ bundle (a point in the Jacobian).

As the authors of Ref. [5] argue, this space is at least birational to a symmetric product of K3 itself. Since the ground states of a supersymmetric sigma model are known to be equivalent to the set of all cohomology classes, it remains to compute the total cohomology space** of this symmetric product. Such a counting is made simple by orbifold techniques [16][17] (believed exact for hyperkahler manifolds), and was computed in Ref. [18]. Let us briefly review this computation.

The cohomology of the symmetric product is given by the oscillator level g for bosonic and fermionic oscillators α_{-k}^i acting on a Fock vacuum $|0\rangle$, where $k = 1 \dots g$ and $i = 1 \dots \dim H^*(M)$, one for each cohomology element – with the bose/fermi statistics depending on whether it is an odd or even cohomology class. The reason this is so is that the orbifold cohomology contains the S_g -invariant cohomology of $M^{\otimes n}$, i.e. the properly symmetrized polynomials in $\dim H^*(M)$ variables, equivalent to states of the form

$$\alpha_{-1}^{i_1} \dots \alpha_{-1}^{i_g} |0\rangle.$$

** Not to be confused with “total cohomology,” a term used for double complexes.

Here all the oscillators are bosonic for K3 and so the corresponding polynomials are totally symmetric. The twisted sectors, one for each conjugacy class of S_g account for the rest of the oscillators. The conformal weights of the oscillators in the twisted sectors are determined by the order of the cyclic pieces of the permutations (all are factorable into products of cyclic permutations) in a conjugacy class – see [18] for details. By this reasoning, we can compute all the dimensions of the cohomology rings the $\text{Sym}^g(M)$ at once, and organize them in a generating function. The result for K3, which has no cohomology classes of odd degree (and hence no fermionic oscillators), is that

$$\sum_{g=1}^{\infty} q^g \dim H^*(\text{Sym}^g(\text{K3})) = \prod_{g=1}^{\infty} \frac{1}{(1 - q^g)^{24}} = q\eta(q)^{-24}.$$

In fact, BPS states have a much simpler description [8][19], if one appeals to the string duality relating the Type II string on K3 to the heterotic string on T^4 , defined most conveniently via the Narain lattice [20][21]. In the heterotic theory, the BPS states have a simple description [6]: the right-moving oscillators must be in their ground states. The right-moving momentum, p_R , yields the mass; the left oscillator number gives the topological type (p^2) of the BPS state. In the Type II language we have that the self-intersection number, equivalently the topological genus, of the curve is equal to the left oscillator level. But since left-movers are just free bosons, the number d_g of states at oscillator level g is just the coefficient of q^g in the partition function of the bosonic string (ignoring the $q^{-c/24}$ piece). This is the same as the result stated above! (See [19], too.) For example, at oscillator level 3 we have states of the form

$$\alpha_{-1}^i \alpha_{-1}^j \alpha_{-1}^k |0\rangle, \quad \alpha_{-2}^i \alpha_{-1}^j |0\rangle, \quad \alpha_{-3}^i |0\rangle.$$

The number of states of the first type is the number of symmetric polynomials of degree three in twenty-four variables (the transverse modes of the string), or $\binom{24+3-1}{3} = 2600$. Thus $d_3 = 2600 + (24)^2 + 24 = 3200$. Higher levels can be obtained similarly, or by expanding the infinite product in a power series. One finds:

$$\begin{aligned} q(\eta(q))^{-24} &= \prod_{n=1}^{\infty} \frac{1}{(1 - q^n)^{24}} \\ &= \sum_{g=0}^{\infty} d_g q^g \\ &= 1 + 24q + 324q^2 + 3200q^3 + 25650q^4 + 176256q^5 + 1073720q^6 + \dots \end{aligned}$$

and

$$d_g = \chi(\mathcal{M}_g^H). \quad (2.1)$$

These numbers agree precisely with those known by mathematicians for the numbers N_g of rational g -nodal curves in K3, a computation to which we now turn. The numbers N_g for $g > 6$ are not known mathematically.

3. Nodal Curves in Linear Systems

In order to compare with computations done in mathematics, we relate the physical question to an equivalent mathematical one: how many \mathbf{P}^1 's with g double points sit inside K3? Or, how many hyperplanes, among a linear system in \mathbf{P}^g , intersect an embedded K3 in g double points? In order to understand the equivalence, it is helpful to elaborate on what is being asked through a simple example.

Consider a K3 embedded in \mathbf{P}^3 , expressed as the zeros of a degree four homogeneous polynomial (here of Fermat type)

$$X^4 + Y^4 + Z^4 + W^4 = 0,$$

where (X, Y, Z, W) are homogeneous coordinates (the results will be independent of the complex structure of K3, so we choose a simple embedding). A linear system is a set of hyperplanes, or zeros of a linear polynomial. We can parametrize a hyperplane by the complex coefficients $\alpha = (a, b, c, d)$ of the linear polynomial:

$$H_\alpha = \{aX + bY + cZ + dW = 0\} \subset \mathbf{P}^3.$$

Clearly the space of inequivalent α defines a \mathbf{P}^4 , which we call \mathbf{P}^{4*} . Now H_α intersects the K3 along the curve (say, when $d \neq 0$)

$$P_\alpha = d^4 X^4 + d^4 Y^4 + d^4 Z^4 + (-aX - bY - cZ)^4 = 0.$$

The adjunction formula (physicist's version) tells us that this curve has (topological) genus $g = 3$, as expected. However, not all such curves are non-singular. Whether the curve is singular depends on whether the equations

$$P_\alpha = dP_\alpha = 0$$

(here d is exterior differentiation) have simultaneous solutions other than $X = Y = Z = 0$. This clearly depends on the modulus α . In fact, the simultaneous equations can be written as the discriminant locus of a larger equation, whose solutions correspond to the points α describing singular submanifolds. The set of all $\alpha \in \mathbf{P}^{4*}$ such that this intersection is a rational curve with 3 (in general g) nodes (i.e. double points) defines a subset of \mathbf{P}^{4*} . For the case considered, this subset is a finite point set with a number of points N_g to be determined.

Why is this number equal to the dimension of the cohomology space of the moduli space, \mathcal{M}_g^H , of supersymmetric cycles of genus g with line bundles (and given homology type)? First, since all the cohomology elements occur with even dimension (as we saw in the last section), we are simply computing the Euler characteristic of \mathcal{M}_g^H . This was, perhaps, a fortunate coincidence; for generalizations of this problem one would have to determine whether it makes sense to count the BPS states with or without signs to relate to mathematical computations. For example, T^4 has $\chi(T^4) = 0$, but indeed has BPS states; so either the mathematics is more naturally associated to the full cohomology ring, or the would-be invariants are trivial. Happily neglecting such subtleties, we prove equivalence to the Euler characteristic for the case at hand.

Consider a K3 embedded in a \mathbf{P}^g such that L , the pull-back of the hyperplane line bundle has $h^0 = g + 1$, i.e. all the global sections – the homogeneous coordinate functions X^i – are preserved. It is clear here that a choice of global sections of L define the embedding, up to projective transformations. This describes the canonical embedding into $\mathbf{P}(H^0(\text{K3}, L))$. The zeros of the global sections $\sum a_i X^i$ define divisors, equivalent to hyperplanes intersecting the K3. Let Σ be such a divisor, so that $L = [\Sigma]$. The \mathbf{P}^{g*} of such divisors is called the linear system. Describing the hypersurfaces locally as the zeros of a defining polynomial, one finds for the normal bundle, $N_\Sigma = [\Sigma]|_\Sigma$. Then the exact sequence

$$0 \longrightarrow T\Sigma \longrightarrow TK3|_\Sigma \longrightarrow N_\Sigma \longrightarrow 0$$

tells us, upon dualizing (which reverses arrows) and taking the exterior product,

$$K_\Sigma = (K_{\text{K3}} + [\Sigma])|_\Sigma. \tag{3.1}$$

Here K denotes the canonical bundle (the highest exterior power of the holomorphic cotangent bundle) and additive notation is used for tensor products. The Riemann-Roch theorem for surfaces [22] relates the Euler characteristic of a line bundle to properties of the variety:

$$\chi([\Sigma]) = \chi(\mathcal{O}_{\text{K3}}) + ([\Sigma] \cdot [\Sigma] - K_{\text{K3}} \cdot [\Sigma])/2.$$

Here the dot product is integration of the first Chern class of the line bundle. Now $\chi(\mathcal{O}_{K3}) = h^{0,0} - h^{1,0} + h^{2,0} = 2$, and $K_{K3} \cdot [\Sigma] = 0$ since the canonical bundle is trivial. On the left hand side, we have $H^p([\Sigma]) = 0$, $p > 0$, by the Kodaira vanishing theorem, since K_{K3} is trivial and $[\Sigma]$ has global sections. Thus $\chi([\Sigma]) = g + 1$. Switching the sign of $K_{K3} \cdot [\Sigma]$ on the right hand side, and using (3.1), we see

$$K_{\Sigma} \cdot [\Sigma] = 2g - 2,$$

i.e. Σ has topological genus equal to g . As the generic K3 has a one-dimensional Picard group of line bundles (obtained from the embedding), we will count all the holomorphic curves this way; if there were more, there would be another divisor class.

The key point now is that the space \mathcal{M}_g^H , defined as the (compactification of the) space of solutions to the Hitchin equations describing the space of holomorphic curves with choice of flat bundle, fibers over the space of holomorphic curves on K3, with fiber equal to the Jacobian of the curve. The map is simple – just forget about the bundle information:

$$\begin{array}{ccc} \text{Jac}(\Sigma_{\alpha}) & \longrightarrow & \mathcal{M}_g^H \\ & & \downarrow \\ & & \mathbf{P}^{g*}, \end{array}$$

where Σ_{α} is the curve described by $\alpha \in \mathbf{P}^{g*}$. A non-singular curve will have non-singular Jacobian, isomorphic to a torus T^{2g} . A singular curve, say one described by $\alpha_i \in \mathbf{P}^{g*}$, will have a fiber F_i equal to a “singular,” or more precisely, compactified, Jacobian. Therefore to compute the Euler characteristic, we can use the simple fact that the Euler characteristic is additive. Let $S = \bigcup S_i$ be the union of components $S_i \subset \mathbf{P}^{g*}$ describing singular curves. Then we have

$$\begin{aligned} \chi(\mathcal{M}_g^H) &= \chi(\mathbf{P}^{g*} \setminus S) \cdot \chi(T^{2g}) + \sum_i \chi(S_i) \cdot \chi(F_i) \\ &= 0 + \sum_i \chi(S_i) \cdot \chi(F_i) \end{aligned} \tag{3.2}$$

Generically, curves with $g - k$ holes and k nodes exist on a subvariety of codimension k in \mathbf{P}^{g*} . The \mathbf{P}^1 's with nodes exist on a finite point set.

Let us consider what types of singular curves will result. The fibers over the singular curves result from a Mumford compactification of the Hitchin space (which is the total family of Jacobians of the holomorphic curves) and correspond to compactifications of the generalized Jacobian of singular spaces. These sorts of Jacobians were considered in [23]. We shall work only with nodal singularities or double points (which look locally like

$y^2 = x^2$). The problem of compactifying the generalized Jacobian over higher singularities is poorly understood at this time; we hope that a better mathematical understanding will corroborate our conjecture that they have zero Euler characteristic, which we compute here for the more common nodal singularities. We now show that the Euler characteristic is zero for all compactified Jacobians over Riemann surfaces with nodes, except for those describing curves of genus zero with g double points. These Jacobians all have Euler characteristic equal to one.***

Consider first a \mathbf{P}^1 with one node; call it X . Its normalization is a \mathbf{P}^1 with no nodes. One constructs a flat bundle over X by considering a flat bundle over \mathbf{P}^1 and then identifying the fibers over two points sitting over the double point. The identification of the fibers gives a \mathbf{C}^* ambiguity, so that the generalized Jacobian, J , sits inside a sequence

$$0 \longrightarrow \mathbf{C}^* \longrightarrow J \longrightarrow \{\text{pt}\} \longrightarrow 0,$$

where $\{\text{pt}\}$ is just a point, representing the Jacobian of \mathbf{P}^1 . The compactification \overline{J} of J is a one-point (one-stratum) compactification corresponding to the (unique up to isomorphism) line bundle over a blowup, up to isomorphism, obtained by an exceptional divisor intersecting the normalization at the two points to be identified. This line bundle must be degree one on the exceptional divisor, and degree -1 on the normalization – the total degree must be zero, as we are describing a Jacobian. There is one such line bundle, since the normalization \mathbf{P}^1 has trivial Jacobian. The result is a one-point compactification of \mathbf{C}^* , which, coincidentally, can be thought of again as \mathbf{P}^1 with a node. What is important to us is the stratification. Let us calculate $\chi(\overline{J})$ by summing up the pieces. First we have an open set $\mathbf{C}^* \times \{\text{pt}\}$, representing a line bundle over a point (in general, this point will be replaced by the Jacobian of the normalization). This has Euler characteristic zero. Then we have the single stratum which is equal to a point, with Euler characteristic equal to one. The total Euler characteristic is one.

For $d > 0$ nodes on a \mathbf{P}^1 the proof is similar, with \mathbf{C}^* getting replaced by $(\mathbf{C}^*)^d$, corresponding to gluing choices of d pairs of points on the normalization of the curve. A dense open subset of \overline{J} is always obtained as a $(\mathbf{C}^*)^d$ bundle over the Jacobian of the normalization. This has Euler characteristic zero. The lowest stratum is a point corresponding to a *unique* line bundle with prescribed degrees over the blowup with exceptional divisors intersecting paired points – which has Euler characteristic one. For higher genus Riemann

*** We thank L. Caporaso for explaining the compactified Jacobians and their stratification.

surfaces with nodes, there is again a stratification, and the proof is again by induction. Zero always results, for the strata are either spaces of line bundles over Riemann surfaces – in which case the Jacobian is a non-trivial torus, yielding zero Euler characteristic – or are higher-genus surfaces with fewer singularities, which will fall under the inductive hypothesis. \mathbf{P}^1 is special because it has a unique flat bundle. For higher types of singularities, which can occur, we expect a zero contribution.

Summarizing, we found for nodal singularities

$$\sum_i \chi(S_i) \cdot \chi(F_i) = \sum_{\substack{\text{genus } p > 0, \\ g-p \text{ nodes}}} \chi(S_j) \cdot \chi(\overline{J}_{S_j}) + \sum_{\substack{\text{genus } 0, \\ g \text{ nodes}}} \chi(\{\alpha_j\}) \cdot \chi(\overline{J}_{\alpha_j}),$$

where in the first term $\chi(\overline{J}_{S_j}) = 0$ for all types of singular curves labeled by S_j , and $\chi(\{\alpha_j\}) = \chi(\overline{J}_{\alpha_j}) = 1$ for the singular curves in the second sum. Thus, recalling (2.1) and (3.2), we have

$$\chi(\mathcal{M}_g^H) = N_g = d_g. \quad (3.3)$$

4. Conclusions and Prospects

We hope this argument leads to further mathematical confirmation of the observations in this paper. In particular, we conjecture that $N_7 = 5930496$, $N_8 = 30178575$, and so on. These numbers, defined rigorously, can be considered as tests of string duality or of the strength of the arguments in deriving the effective theories of D-branes. That agreement has, to date, been achieved up to $g = 6$ is already quite striking. Note that for $g \leq 2$ although there is no embedding of the K3 in projective space \mathbf{P}^g , it makes perfect sense to consider the spaces of holomorphic curves of genus $g \leq 2$. The $g = 1$ case, for example, corresponds to the map $\text{K3} \rightarrow \mathbf{P}^1$, viewing K3 as an elliptic fibration over \mathbf{P}^1 with 24 singular fibers. For $g \geq 7$, we eagerly await mathematical computations.

Unfortunately, both the physical and mathematical pieces of the puzzle are quite difficult. What made this problem tractable was 1) the identification of the moduli space around a supersymmetric cycle, made possible essentially by K3 being a toric fibration; 2) the simple dual string computation; 3) the relation, in two dimensions, between supersymmetric cycles and holomorphic maps. As K3 and T^4 are the only viable four-dimensional compactifying spaces, generalizations of the above lead us to higher-dimensional spaces. For Calabi-Yau compactifications, special Lagrangian manifolds are

(real) three-dimensional, and so we lose any connection to holomorphicity. It is unlikely, therefore, that the computation has a simple algebro-geometric description. Supersymmetric four-cycles (not special Lagrangian) in Calabi-Yau's may have a holomorphic interpretation, but tests of string duality and quantum mirror symmetry [24][25] seem to require a better mathematical understanding of special Lagrangian embeddings.

Acknowledgements

We would like to thank L. Caporaso, X. Chen, J. Harris, B. Lian, A.-K. Liu, S. Sroek, and C. Vafa for discussions. This work is supported by grant DE-F602-88ER-25065.

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